However, the capillarity of the soil has the greatest effect on the width of the irrigated area. It is evident from the last division of Table 1 that L increases by a factor of 27.2 in the first series with an increase in the parameter h_V from 0.1 to 0.89. It should be noted that at $h_V \approx 0$ and $h_V \approx T - H$, the radius of capillary flow exceeds the height of capillary rise h_V. Meanwhile, the largest difference is reached at values of h_V close to T − H. Thus, in the case $h_V = 0.89$, L = 2.7132 and, thus, $L/h_V = 3.0$. As a result, the substantial value of horizontal absorption noted in [1, 5] — even for low-capillarity soils — is confirmed to exist. Calculations showed that an increase in the head H leads to an even greater spread. For example, in the second series of Table 1 with $h_V = 0.69$, we obtain $L/h_V = 3.7$. As regards the flow rate, it changes by 37 and 27% for the values of h_V shown in the third division of Table 1.

Let us follow the effect of the depth of the high-permeability layer at D = 0.3 and $h_V = 0.1$, fixing the value $T - H - h_V = 0.8$. The results of the calculations are shown in Table 2. It is evident that the capacity of the layer nearly ceases to affect the radius of capillary flow at T > 5. At large values of T, the last two values of L differ from one another by no more than 1.5%. The effect of T on flow rate turns out to be somewhat greater; the latter can be considered negligible at T > 7.

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FINITE RATE OF RADIANT HEAT TRANSFER IN A GRAYBODY IN THE PRESENCE OF HEAT SOURCES (SINKS)

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Different intensive heat-transfer processes which take place in the presence of significant temperature gradients are currently being discussed in the literature. The study of such processes is complicated by the need to make allowance for the variable thermophysical properties of the substance being investigated. This applies in particular to radiant heat transfer. Here, the main characteristic of the substance is the mean free path of the radiation, which depends appreciably on temperature [1].

Radiant heat transfer is described by nonlinear integrodifferential equations in accordance with the nonlocal character of interaction of radiation with a substance [1, 2]. In many important cases, it is sufficient to limit the investigation to a graybody approximation [1], assuming that the absorption coefficient is independent of the spectral composition of the radiation. In the event of planar symmetry, the integrodifferential equation has the following form in dimensionless variables [1, 2] in the presence of heat sources (sinks)

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$$\frac{\partial E}{\partial t}=\varkappa^2 k(U-T^4)+Q,\quad U=\frac{1}{2}\int\limits_{-1}^{1} I d\mu. \tag{0.1}$$

Here, $E(T) \geq 0$ $[E(0) = 0]$ is the specific energy of the substance, being a monotonically increasing function of temperature; $T(x, t) \geq 0$ is the temperature of the substance; $x \in R^1$ is the space coordinate along which heat is transmitted; $t > 0$ is time; $U(x, t) \geq 0$ is the volume density of radiant energy; $k = k(T)$ is the coefficient of absorption of radiation by the substance $[0 < k(T) < \infty$ at $0 < T < \infty$, $k(0) = 0]$; $Q(T)$ is a function of the heat sources (sinks) $[Q(0) = 0]$; $\kappa^2 = (L/\ell)^2$; L is the characteristic dimension of the region heated by radiation; ℓ is the characteristic mean free path of the radiation; $I(x, t, \mu)$ is the intensity of the radiation, determined from the formula

$$I=\begin{cases} I_+=\dfrac{\varkappa^2}{\mu}\displaystyle\int\limits_{-\infty}^{x} k[T(\xi,t)]\,T^4(\xi,t)\exp\left[-\frac{|P(x,\xi)|}{\mu}\right]d\xi, & \mu>0,\\ I_-=\dfrac{\varkappa^2}{\mu}\displaystyle\int\limits_{\infty}^{x} k[T(\xi,t)]\,T^4(\xi,t)\exp\left[-\frac{|P(x,\xi)|}{\mu}\right]d\xi, & \mu<0,\end{cases} \tag{0.2}$$

$$P(x,\xi)=\varkappa^2\int\limits_{x}^{\xi} k[T(\varepsilon,t)]\,d\varepsilon,$$

where $\mu = \cos\theta_*$; θ_* is the angle between the direction of the x axis and an arbitrarily chosen direction of radiation $(0 \leq \theta_* \leq \pi)$. It should be noted that Eq. (0.2) is valid if the condition of finiteness of the temperature increase at $|x| \to \infty$ is satisfied [1], thus ensuring the existence of improper integrals in definitions (0.2) [for example, $T(x, t) < M$, $M = \text{const}$, $0 < M < \infty$].

The passage to the limit $\kappa^2 \to \infty$ in (0.1) signifies the transition to the approximation of radiant heat transfer. In this approximation, the problem reduces to the analysis of a quasilinear differential equation of the parabolic type [3]. This equation turns out to be nonlinear, even if the absorption coefficient $k(T) = \text{const} > 0$. It was found in [3] that intensive local heat release may result in the transfer of heat in the form of a thermal wave. The front of this wave strictly delineates the boundary between the regions of cold and hot substance. In physical terms, the presence of the frontal surface of the thermal wave means that the velocity of propagation of thermal disturbances is finite. The effect of sources (sinks) on the propagation of thermal waves has been studied in many different investigations that cannot be discussed here. We note only [4], which can serve as an introduction to the present problem.

The sources and sinks in radiant heat transfer may be different in nature. The release or removal of heat is possible in exothermic and endothermic chemical reactions and phase transformations in matter [1, 5, 6]. The cooling of a substance due to volumetric de-excitation is also to be considered. This phenomenon is connected with the existence of "transparent windows" in a cold substance, i.e., frequency intervals for which the mean free path of the radiation in the cold substance is large. Along with these factors, it is possible to approximately account for energy losses by the substance by introducing an appropriate sink.

The author of [8] studied Eq. (0.1) with $Q \equiv 0$ and obtained necessary conditions for the existence of frontal surfaces [the main requirement being the condition of unboundedness of the function $k(T) \to \infty$ at $T \to 0$]. Below, we analyze the effect of heat sources (sinks) $Q \not\equiv 0$ on the problem.

1. Simple Wave. First we will examine a particular solution to Eq. (0.1) of the simple-wave type. Let $T = T(\eta)$, $I = I(\mu, \eta)$ $(\eta = x - vt,\ v = \text{const} \neq 0)$. Then Eq. (0.1) reduces to the form

$$-v\frac{dE}{dz}=\frac{1}{2}\int\limits_{-\infty}^{\infty} T^4 W_1(|z-\xi|)\,d\xi-T^4+\frac{Q}{\varkappa^2 k}. \tag{1.1}$$

Here, $W_n = \int\limits_1^\infty \exp(-y\tau)\tau^{-n}d\tau$; $n = 0, 1, 2, \ldots$ is an integral exponential function [9]; $z = \kappa^2 \int k d\eta$ is the optical thickness of the substance [for the sake of brevity, the arguments are omitted in (1.1) and the subsequent discussion). Equation (1.1) must be augmented by boundary conditions at $z = \infty$ or $z = -\infty$. For the sake of determinateness, we put

$$T = 0, \; I = 0 \text{ at } z = \infty. \tag{1.2}$$

Integrating (1.1) with allowance for (1.2), we obtain the equation

$$vE = \frac{1}{2} \int_{-\infty}^{\infty} T^4 \operatorname{sgn}(z-\xi) W_2(|z-\xi|)\, d\xi + \int_z^{\infty} \frac{Q}{\varkappa^2 k}\, d\xi. \tag{1.3}$$

Integrating (1.3) twice, we obtain the relation

$$v\int_z^{\infty}\int_{\xi}^{\infty} E\, d\varepsilon\, d\xi = \frac{1}{2}\int_{-\infty}^{\infty} T^4 \operatorname{sgn}(z-\xi) W_4(|z-\xi|)\, d\xi + \tag{1.4}$$

$$+ \frac{1}{3}\int_z^{\infty} T^4\, d\xi + \int_z^{\infty}\int_{\xi}^{\infty}\int_{\zeta}^{\infty} \frac{Q}{\varkappa^2 k}\, d\varepsilon\, d\zeta\, d\xi.$$

Using familiar inequalities between integral exponential functions [9], we can write $W_4(|z - \xi|) = \alpha(|z - \xi|)W_2(|z - \xi|)$ $\alpha(|z - \xi|) \in [1/3, 1]$. Allowing for this relation, using the mean-value theorem, and employing (1.3), we write Eq. (1.4) as

$$v\int_z^{\infty}\int_{\xi}^{\infty} E\, d\varepsilon\, d\xi = \delta\left(vE - \int_z^{\infty} \frac{Q}{\varkappa^2 k}\, d\xi\right) - \frac{1}{3}\int_z^{\infty} T^4\, d\xi + \tag{1.5}$$

$$+ \int_z^{\infty}\int_{\xi}^{\infty}\int_{\zeta}^{\infty} \frac{Q}{\varkappa^2 k}\, d\varepsilon\, d\zeta\, d\xi, \quad \delta(z) \in [1/3, 1].$$

It should be noted that Eq. (1.5) is convenient for asymptotic analysis of the solution at $z \to \infty$, since it involves only integration within the limits of the ray $\xi \in [z, \infty]$.

We are interested in the frontal solutions of Eqs. (0.1) or (1.5). Such solutions are characterized by the presence in the plane x, t of interfaces (solution fronts) which strictly delimit the region $\Omega^+ = \{(x, t) : T(x, t) > 0\}$ and the background $\Omega_0 = \{(x, t) : T(x, t) \equiv 0\}$ [8]. The complete solution of Eq. (1.5) can apparently be obtained only by numerical methods.

It follows from physical considerations that the necessary conditions for continuity of the temperature T, volume density of radiant energy U, and intensity I must be satisfied at the points of the curve $x = x_f(t)$ (wavefront).

We will examine the solution $T(\eta)$ of Eq. (1.5) near the front $\eta = \eta_f(t)$, assuming that the front exists. For the sake of definiteness, we put $T(\eta) > 0$ at $\eta > \eta_f$ and $T(\eta) \equiv 0$ at $\eta \geq \eta_f$; we also let $E = T$, $k = T^{-\gamma}$, $Q = qT^{\beta}$, where γ, $\beta > 0$, and q are arbitrary constants. Meanwhile, it follows from physical considerations that $\gamma < 4$ [1]. (Here, it is appropriate to note that for volumetric de-excitation phenomena, the source can be considered proportional to T^{β}, $\beta \geq 1$. If the "transparent windows" correspond to the Rayleigh–Jeans frequency region for the characteristic heating temperature, then $\beta \approx 1$. This is the very situation realized in the case of a thermal explosion in air [1]. Other situations are also possible [10].)

We will assume that the asymptotic representation of $T(\eta)$ at $\eta \to \eta_f - 0$ is determined by the expression [4]

$$T(\eta) \sim \theta(\eta_f - \eta)^{\omega}, \; \omega, \theta = \text{const} > 0. \tag{1.6}$$

Returning to the physical variable η in Eq. (1.5) and inserting (1.6) into this equation, we obtain the algebraic equation

$$va_1(\eta_f - \eta)^{\omega(1-2\gamma)+2} = v\delta_* \theta(\eta_f - \eta)^{\omega} + qa_2(\eta_f - \eta)^{\omega\beta+1} + \tag{1.7}$$

$$+ a_3(\eta_f - \eta)^{\omega(4-\gamma)+1} + qa_4(\eta_f - \eta)^{\omega(\beta-2\gamma)+3},$$

where

$$\delta_* = \delta(\infty); \; a_1 = \frac{\varkappa^4 \theta^{1-2\gamma}}{[\omega(1-\gamma)+1][\omega(1-2\gamma)+2]}; \quad a_2 = -\frac{\delta_* \theta^{\beta}}{\omega\beta+1};$$

$$a_3 = \frac{\varkappa^2\theta^{4-\gamma}}{3[\omega(4-\gamma)+1]};\quad a_4 = \frac{\varkappa^4\theta^{\beta-2\gamma}}{(\omega_\beta+1)[\omega(\beta-\gamma)+2][\omega(\beta-2\gamma)+3]}.$$

The character of the asymptotic solution $T(\eta)$ of Eq. (1.5) near the front is determined by the conditions of solvability of Eq. (1.7); for this, it is necessary that at least two of the exponents in (1.7) coincide and that the remaining exponents be greater than these two. By analyzing different variants of relations between the exponents with $(\eta_f - \eta)$ in individual terms of (1.7), we find possible values of ω and θ and thus obtain relations for the exponents γ and β. The latter relations determine the structure of the asymptotic representation of the solution in the neighborhood of the front. Thus, it turns out to be possible to classify the frontal solutions.

Let $\beta \geq 1$. In this case, in the asymptotic representation of the solution $\omega = \omega_1 \equiv 1/\gamma$, $\theta = \theta_1 \equiv (\kappa^2\gamma/\sqrt{\delta_*})^{1/\gamma}$. We can use Eq. (1.1) to obtain an asymptotic representation for the volume density of radiant energy $U \sim -(v/\kappa^2 k)dT/d\eta$. The condition $U \geq 0$ requires that $v > 0$, which means physically that only a heating wave exists; here, asymptotic representation (1.6) is independent of the action of the heat sources $q > 0$ or heat sinks $q < 0$. (Similar results were obtained in [8] with $q \equiv 0$.) Thus, if $\beta \geq 1$, then the effect of sources (sinks) on the formation of the front of the thermal wave is negligible. Such heat sources (sinks) will be called "weak" sources (sinks).

Now we put $0 < \beta < 1$. Analysis of Eq. (1.7) shows that, in this case, sources (sinks) may have a decisive effect on the formation of the frontal surface. We will call sources (sinks) "strong" when $0 < \beta < 1$. Along with the asymptote established by the exponent $\omega = \omega_1$, there is yet another possibility: $\omega = \omega_2 \equiv 1/(1-\beta)$. Asymptotic representation (1.6), with $\omega = \omega_1$ at $0 < \beta < 1$, depends on the sum $\gamma + \beta$; here, the expression for the coefficient θ in (1.6) has the form

$$\theta = \begin{cases}\theta_1, & \gamma+\beta > 1,\\ \theta_2, & \gamma+\beta < 1\end{cases}$$

$(\theta_2 \equiv \{\kappa^2\gamma/[\sqrt{\delta_*}(\gamma + \beta)]\}^{1/\gamma})$. The condition $U \geq 0$ requires that $q < 0$, $v > 0$.

Another construction of asymptotic representation (1.6), with $\omega = \omega_2$, can be realized for all values $0 < \gamma < 4$. Then the multiplier θ is determined from (1.7): $\theta = \theta_3 \equiv [q(1-\beta)/v]^{1/(1-\beta)}$. It follows from this expression that sign v = sign q, i.e., the asymptotic representation being examined is possible with the action of either sources ($q > 0$, $v > 0$) or sinks ($q < 0$, $v < 0$). Thus, both heating ($v > 0$) and cooling ($v < 0$) waves are possible.

It is interesting to note that for strong sources (sinks) a front appears even for $\gamma = 0$, i.e., in the absence of degeneracy of the absorption coefficient at $T \to 0$: $0 < k(0) < \infty$.

For greater clarity, Table 1 shows possible regimes of localization of thermal disturbances with the corresponding values of ω and θ in asymptotic representation of the solution $T(\eta)$ (1.6) as a function of the parameters γ and β. For completeness, the Table also shows the results for the case $\gamma + \beta = 1$.

2. Example of an Analytically Closed Solution. We noted above that serious complications are encountered when an attempt is made to study Eqs. (1.1) or (1.5) analytically. Thus, to confirm the conclusions made above, we will present an example in which the solution of Eq. (0.1) can be obtained analytically in closed form. In accordance with [11], we put $E = T^4$. We can also apply $Q = \varkappa^2 k(aT^4 + b\,dT^4/dz)$ (a, b are arbitrary constants). Then the variables are separated in Eq. (0.1). As a result, we find that the temperature $T(\eta)$ is determined from the equation

$$\eta = \frac{4}{\varkappa^2 \nu}\int_0^T \frac{dT}{kT} + \eta_f,\quad \eta_f = \text{const},\quad |\eta_f| < \infty,$$

while the intensity I is found from the formula $I = T^4/(\nu\mu + 1)$, where the constant for separation of the variables $\nu \in (-1, 0)$ determines the velocity of the thermal wave

$$v = \frac{1}{\nu}\left[\left(1 - \frac{1}{2\nu}\ln\frac{1+\nu}{1-\nu}\right) - (a + \nu b)\right].$$

The sign of the function $Q(T)$ coincides with the sign of the sum $c = a + \nu b$. At $c > 0$, only a heating wave is possible ($v > 0$). At $c < 0$, both a heating wave ($v > 0$) and a cooling

TABLE 1

v		$\gamma+\beta<1$		$\gamma+\beta=1$		$\gamma+\beta>1$			
		$\beta<1$		$\beta<1$		$\beta<1$		$\beta\geqslant1$	
		$q>0$	$q<0$	$q>0$	$q<0$	$q>0$	$q<0$	$q>0$	$q<0$
>0	ω	ω_2	ω_1	ω_2	—	ω_2	ω_1	ω_1	ω_1
	θ	θ_3	θ_2	θ_3	—	θ_3	θ_1	θ_1	θ_1
<0	ω	—	ω_2	—	ω_2	—	ω_2	—	—
	θ	—	θ_3	—	θ_3	—	θ_3	—	—

wave ($v < 0$) are possible, which confirms the conclusions reached above (see Table 1).

3. Asymptotic Representation for the Temperature Near the Front of a Thermal Wave in the General Case. The conditions of the existence of the front obtained above for a simple wave can also be formulated as necessary conditions in the general case of radiant heat transfer in a graybody.

Let there be the surface $x = x_f(t)$, $\dot{x}_f \equiv dx_f/dt \not\equiv 0$, this surface being a front, i.e., $T(x, t) > 0$, $x < x_f(t)$ and $T(x, t) \equiv 0$, $x \geq x_f(t)$. Following [12], we differentiate the condition $E(x_f(t), t) = 0$ with respect to time $\partial E/\partial t + \dot{x}_f \partial E/\partial x = 0$, $x = x_f(t)$. We assume that this equality is satisfied asymptotically at $x \to x_f(t) - 0$ and, having replaced the derivative $\partial E/\partial t$ in (0.1), we introduce the relation

$$-\dot{x}_f \partial E/\partial t = \varkappa^2 k(U - T^4) + Q, \quad x \to x_f - 0. \tag{3.1}$$

It is evident from a comparison of (3.1) and (1.1) that they coincide if we put $v = \dot{x}_f$, $\eta = x - x_f$. Thus, the results obtained above for the case of the existence of a front for a simple wave apply fully to the case of an arbitrarily moving frontal surface $x = x_f(t)$, $\dot{x}_f \neq 0$.

As a result, the presence of heat sources (sinks) may have a significant effect on the formation and propagation of fronts of thermal disturbances in radiant heat transfer.

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